From this we can evaluate the reverberation time, i.e. the time interval T, in which the reverberating sound energy reaches one millionth of its initial value:

$$T = -\frac{24V\ln 10}{cS\ln(1-\bar{\alpha})} \tag{V.22}$$

We can complete this formula by taking into account, as in Equations (IV.4) to (IV.9), the attenuation constant m of air which is responsible for the attenuation of sound during its free propagation and by inserting the numerical value of the sound velocity of air:

$$T = -0{\cdot}163\frac{V}{S\ln(1-\bar{\alpha}) - 4mV} \tag{V.23}$$

with

$$\bar{\alpha} = \frac{1}{S}\sum S_i\alpha_i \tag{V.23a}$$

where we have already generalised Equation (V.20*a*) for any number of different portions of wall. In this formula, which is probably the most important relation of room acoustics, all lengths have to be expressed in metres; T is measured in seconds.

Equation (V.23) together with (V.23*a*) is known as Eyring's reverberation formula, although it has been derived independently by Norris as well as by Schuster and Waetzmann.

For many practical purposes it is safe to assume that the average absorption coefficient $\bar{\alpha}$ is small compared with unity. Then the logarithm in Equation (V.23) can be expanded into a series and all terms of higher than the first order in $\bar{\alpha}$ may be neglected. This results in a reverberation formula originally derived by Sabine:

$$T = 0{\cdot}163\frac{V}{S\bar{\alpha} + 4mV} \tag{V.24}$$

For small rooms the term $4mV$ related to air absorption can be neglected.

V.4 *Other Reverberation Theories, Monte-Carlo Computations*

If, in our derivation of Equation (V.20), we had considered the quantities NS_1/S and NS_2/S not as mean values of a probability

distribution but instead as exact numbers of collisions with the wall portions S_1 and S_2, then Equation (V.19) would also be the exact expression for the reverberant energy after a total of N reflections. It can equally well be written in the following way:

$$E(t) = E_0\exp(-N\bar{\alpha}') = E_0\exp\left(-\frac{cS}{4V}\bar{\alpha}'t\right) \tag{V.25}$$

with the average 'absorption exponent':

$$\bar{\alpha}' = -\frac{1}{S}\sum_i S_i\ln(1-\alpha_i) \tag{V.25a}$$

The resulting equation

$$T = 0{\cdot}163\frac{V}{S\bar{\alpha}'} \tag{V.26}$$

which could again be completed by taking into account the air attenuation by adding a term $4mV$ to the denominator, is known as Millington's formula. It differs from Equation (V.23) only in the manner in which the absorption coefficients of the various portions of wall are averaged: here the average absorption coefficient is replaced by the average absorption exponent.

The application of Equations (V.26) and (V.25*a*) has a strange consequence: let us suppose that a room has a portion of wall, however small, with the absorption coefficient $\alpha_i = 1$. It would make the average (V.25*a*) infinitely large and hence the reverberation time evaluated by Equation (V.26) would be zero. This is obviously an unreasonable result and thus the Millington formula is false.

The incorrect averaging rule of Equation (V.25*a*) was the result of replacing a probability distribution by its mean value. However, in the derivation of Equation (V.21) we have practised a similar simplification in that we have replaced the actual number of reflections in the time t by its average $\bar{n}t$. For a more correct treatment we ought to introduce the probability $P_t(N)$ of exactly N wall reflections occurring in a time t and to calculate $E(t)$ as the expectation value of Equation (V.20) with respect to this probability distribution:

$$E(t) = E_0\sum_{N=0}^{\infty} P_t(N)\exp[N\ln(1-\bar{\alpha})] \tag{V.27}$$

In order to derive an expression for $P_t(N)$, we assume, as before, that the room has a diffuse sound field which guarantees that the fate of one sound particle is representative of all the others. After N reflections, the total distance covered by a certain particle is

$$ct = l_1 + l_2 + \cdots + l_N$$

where the l_i's are individual path lengths distributed according to an unknown law. We need not know this distribution exactly, since we shall invoke the central limit theorem of probability theory which states that the distribution of the total distance ct approaches a Gaussian law if N is sufficiently high and if the single path lengths do not depend on each other. (In practice it is sufficient to take N as being larger than about 3 or 5.) Furthermore, the mean value of ct becomes $N\bar{l}$ while its variance is $N\bar{l}^2\gamma^2$. Thus, we obtain the probability of a time t needed for exactly N reflections:

$$P_N(t) = \frac{c}{(2\pi N)^{1/2}\gamma \bar{l}} \exp\left(-\frac{(ct - N\bar{l})^2}{2N\gamma^2\bar{l}^2}\right) \qquad \text{(V.28)}$$

where γ^2 is the 'relative variance' of the path length distribution already defined in Equation (V.16). (Examples of path length distributions are represented in Fig. V.5.)

Equation (V.28) is at the same time—apart from some constant factors—the probability $P_t(N)$ of exactly N reflections in a prescribed time interval t, where N is not necessarily integer. For ct not to be too small, its value depends highly on the N in the numerator of the exponent, but only slightly on the N's in the denominators. Therefore we can replace the latter ones by their average values at time t, that is by $\bar{n}t = ct/\bar{l}$:

$$P_t(N) \sim \frac{1}{(t)^{1/2}} \exp\left(-\frac{(ct - N\bar{l})^2}{2ct\bar{l}\gamma^2}\right) \qquad \text{(V.29)}$$

where unimportant constants have been omitted. Thus, our distribution is Gaussian too with respect to the variable N. Next we have to insert $P_t(N)$ into Equation (V.27). By replacing the summation by an integration, we obtain after an additional insignificant simplification:

$$E(t) \approx E_0 \exp\left(-\frac{cS}{4V}\alpha'' t\right) \qquad \text{(V.30)}$$

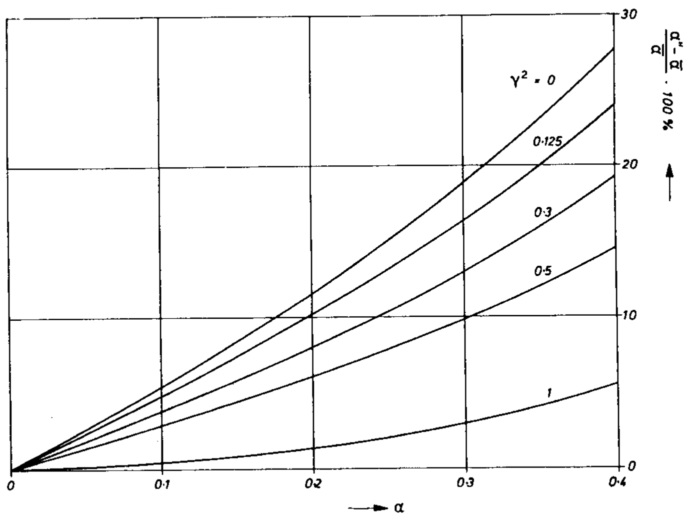

Fig. V.6. Relative difference between α'' and $\bar{\alpha}$ in percentage (see Equation (V.32)).

and for the reverberation time:

$$T = 0{\cdot}163 \frac{V}{S\alpha''} \qquad \text{(V.31)}$$

where the quantity α'' characteristic for the absorption is given by

$$\alpha'' = -\ln(1 - \bar{\alpha})\left(1 + \frac{\gamma^2}{2}\ln(1 - \bar{\alpha})\right) \qquad \text{(V.32)}$$

In Fig. V.6 the relative difference between α'' and $\bar{\alpha}$ in percentage is plotted as a function of α for various parameters γ^2. The curve $\gamma^2 = 0$ corresponds to the Eyring formula (V.22) or (V.23). It is only valid for all paths having exactly the same length, i.e. for a one-dimensional room. For $\gamma^2 \neq 0$, α'' is smaller than $-\ln(1 - \bar{\alpha})$. Hence the reverberation time is longer than that evaluated by the Eyring formula.

With the aid of Equation (V.32) we are able to take account of the fact that in every real room the path lengths between successive wall reflections of a sound particle differ within certain limits. Unfortunately, the relative variance γ^2 of the path length distribution can be calculated directly only for a few room shapes of high symmetry. For other enclosures, it is convenient to evaluate γ^2 by simulating the sound propagation with a digital computer. This is done by assuming a sound particle which leaves its origin (the sound source) under a specified direction. When it hits a wall, its direction is changed and must be recalculated, assuming either specular or diffuse reflection. In the latter case, this is done with the aid of random numbers generated such as to guarantee that the reflection angle is distributed according to Equation (V.9). This process is continued over many reflections, at the same time, the path lengths between the reflections are calculated and classified. Simulation procedures of this kind involving random numbers are known as Monte-Carlo methods.

Values of γ^2 evaluated in this way for rectangular rooms have already been presented in Table V.1. They are smallest for cubical or nearly cubical rooms and become larger than 0·5 only in very long rooms or rooms with low ceiling height. On the whole γ^2 does not strongly depend on the shape of the room. For this reason these values can be applied to rooms of other than rectangular shape, provided that their shapes are not too different from those of rectangular rooms.

For rooms with suspended diffusing elements, the distribution of free path lengths is greatly modified by the obstacles, and the same applies to γ^2, but not to the mean free path.[5]

There is still another assumption which we have made in the course of Section V.3 without proving its justification: the assumption that the average number of reflections occurring on a wall element is proportional to its area (*see* Equation (V.17)), or, in other words, that the energy B impinging onto the wall per second and unit area is constant everywhere. This is true only in a few special cases. For a general discussion, we can go back to Equation (V.2), assuming an exponential time dependence of B:

$$B(\mathbf{r}, t) \sim B(\mathbf{r}) \exp\left(-\frac{ct\alpha^*}{\bar{l}}\right)$$

Inserting this law into Equation (V.2) yields

$$B(\mathbf{r}) = \frac{1}{\pi} \iint_S [1 - \alpha(\mathbf{r}')] B(\mathbf{r}') \exp\left(\frac{R\alpha^*}{\bar{l}}\right) \frac{\cos\vartheta \cos\vartheta'}{R^2} \mathrm{d}S'$$

$$= \frac{\exp\alpha^*}{\pi} \iint_S (1 - \alpha) B(\mathbf{r}') \frac{\cos\vartheta \cos\vartheta'}{R^2} \mathrm{d}S'$$

In the latter formula, R in the exponential function has been

approximated by the mean free path $\bar{l}$ neglecting now the effects of different path lengths which have already been discussed above. We integrate this equation once more over the total wall area and change the order of integration on the right hand side. Thus we obtain:

$$\exp\alpha^* = \frac{\iint B(\mathbf{r})\,\mathrm{d}S}{\iint [1 - \alpha(\mathbf{r})] B(\mathbf{r})\,\mathrm{d}S} \qquad \text{(V.33)}$$

This expression shows clearly that constant irradiation strength B leads to

$$\alpha^* = -\ln(1 - \bar{\alpha})$$

i.e. that in this case the Eyring value of the absorption exponent is valid which must be completed, if necessary, by the path length correction of Equation (V.32).

Constant irradiation strength occurs in two cases:

(a) The room has spherical shape. Then $\cos\vartheta \cos\vartheta'/R^2 = \pi/S$.
(b) The absorption coefficient α is constant over the whole wall.

In general, α^* can be larger or smaller than $-\ln(1 - \bar{\alpha})$, depending on the room shape and on the distribution of the wall absorption. This may be illustrated again by Monte-Carlo computations, carried out for rectangular rooms of various shapes.[6] The technique of simulation is similar to that employed for the evaluation of path length distributions as described above. The only difference is, that the wall absorption is taken into account by generating an additional random number y with $0 < y < 1$ for each reflection. This number decides the further fate of a sound particle: only if $y > \alpha$, the particle will be reflected and continue its course through the room until at a particular wall y happens to be smaller than α. In this case the particle is absorbed, and a new particle is started from the source. If this procedure is carried out for a great number of particles, a distribution of particle life times can be evaluated which represents the average decay curve, from which the reverberation time or the 'effective absorption exponent' α^* can be determined.

For the present computation, one of the six room walls was assumed to be totally absorbent ($\alpha = 1$), while the five remaining ones are supposed to reflect all the impinging sound ($\alpha = 0$) in a diffuse manner. The results are plotted in Fig. V.7. The abscissa is the absorption coefficient $\bar{\alpha}$ averaged over all walls, and the ordinate is the effective absorption exponent α^*, divided by its 'Eyring value' $-\ln(1-\bar{\alpha})$. It is seen that the results deviate from the line $\alpha^* = -\ln(1-\bar{\alpha})$ in both directions, and predict in particular for flat rooms with absorbing floor (see right side of the figure) reverberation times which are shorter than those according to Eyring's formula, Equation (V.22). This is of practical interest because virtually all concert halls and many other auditoria are of this general type.

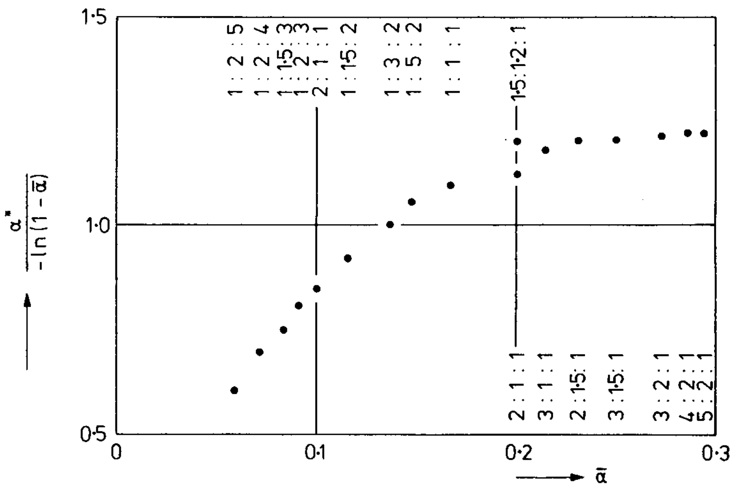

Fig. V.7. Effective absorption exponents α^*, divided by $-\ln(1-\bar{\alpha})$, as obtained from Monte-Carlo computations for rectangular rooms with $\alpha = 1$ for one wall and $\alpha = 0$ for the others. Numbers in the figure indicate the relative room dimensions, the first two numbers refer to the absorbing wall.

The observed differences between α^* and $-\ln(1-\bar{\alpha})$ are caused by the fact that generally not all wall elements are hit by sound particles at equal probability, i.e. by a particular type of diffusion deficiency. They make it evident that diffusion and diffusely reflecting walls are different things, in that the latter do not guarantee perfectly diffuse sound field conditions.

The integral equation of page 114 can be solved for B and finally for α^* by approximation methods, but no closed reverberation formula accounting for the deviations is available so far. The same statement holds for an alternative theoretical approach due to Gerlach and Mellert[7] which is based upon the theory of Markov processes.

V.5 *Steady State Energy Density in a Reverberant Space with Diffuse Sound Field*

Equation (V.21) describes the temporal decay of the sound energy at times during which no sound source is in operation. It can be considered as the energetic response of the room to a short impulse which releases the energy E_0 into the room, i.e. to a sound source which supplies the acoustical power $P(t) = E_0\delta(t)$.

If acoustical power is continuously supplied to a room, we can make use of the relations of Section I.5, according to which the power $P(t)$ can be regarded as a close succession of short energy impulses (*see* Equation (I.36)), while the energetic response of the room after Equation (I.37) is given by

$$E(t) = \int_{-\infty}^{t} P(\tau)\exp\left[\frac{cS}{4V}(t-\tau)\,.\ln(1-\bar{\alpha})\right]\mathrm{d}\tau$$

$$= \int_{0}^{\infty} P(t-\tau)\exp\left[\frac{cS}{4V}\tau\,.\ln(1-\bar{\alpha})\right]\mathrm{d}\tau \qquad \text{(V.34)}$$

the latter expression being the result of a change in the integration variable.

For a constant source power P, we can carry out the integration and obtain the stationary sound energy in the room:

$$E = -\frac{4PV}{cS\,.\ln(1-\bar{\alpha})} \qquad \text{(V.35)}$$

or the energy density:

$$w = -\frac{4P}{cS\,.\ln(1-\bar{\alpha})} \qquad \text{(V.36)}$$

If we want to exclude the contribution of the direct sound in order to obtain the energy density of the reverberant sound field only, we can replace the lower limit of the second integral in Equation (V.34) by $\bar{l}/c = 1/\bar{n}$, the average time for a sound particle to reach the observation point via one wall reflection. This yields:

$$w_r = -\frac{4P}{cS}\frac{1-\bar{\alpha}}{\ln(1-\bar{\alpha})} \qquad \text{(V.36}a\text{)}$$

An alternative derivation starts from Equation (V.20) which